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On Equivalence of Moduli of Smoothness

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It is known that if $f \in \mathbf{W}_p^k$, then $\omega_m(f, t)_p \leqslant t\omega_{m-1}(f', t)_p \leqslant \cdots$. Its inverse with any constants independent of f is not true in general. Hu and Yu proved that the inverse holds true for splines S with equally spaced knots, thus $\omega_m(S, t)_p \sim t\omega_{m-1}(S', t)_p \sim t^2\omega_{m-2}(S'', t)_p \cdots$. In this paper, we extend their results to splines with any given knot sequence, and further to principal shift-invariant spaces and wavelets under certain conditions. Applications are given at the end of the paper.

Key Words: modulus of smoothness; splines; shift-invariant spaces; wavelets; degree of approximation; convex approximation.

1. INTRODUCTION

Let A be an interval of any of the forms $[a, b]$, $\mathbf{R} := (-\infty, \infty)$, $(-\infty, b]$, or $[a, \infty)$. Throughout this paper we denote by $\mathbf{L}_p(A)$ the usual $\mathbf{L}_p(A)$ space for $1 \leqslant p < \infty$, and $\mathbf{C}(A)$ for $p = \infty$. Let Δ_t^k be the kth forward difference operator defined by

$$\Delta_t^k(f, x) := \begin{cases} \sum_{j=0}^{k} (-1)^{k+j} \binom{k}{j} f(x + jt), & \text{if} \quad x, x + kt \in A, \\ 0, & \text{otherwise} \end{cases}$$

and for any $f \in \mathbf{L}_p(A)$ let

$$\omega_k(f, A, t)_p = \omega_k(f, t)_p := \sup_{0 \leqslant u \leqslant t} \|\Delta_u^k f\|_{\mathbf{L}_p(A)}, \qquad t \geqslant 0,$$

be the usual kth modulus of smoothness of f, with $\omega_0(f, t)_p$ understood as $\|f\|_{\mathbf{L}_p(A)}$. Unless we work on subintervals of A, we will very often omit the interval in the notation. If $f \in \mathbf{W}_p^k(A)$, the Sobolev space of functions on A

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such that $f^{(k-1)}$ is absolutely continuous and $f^{(k)} \in \mathbf{L}_p$, it is well known that

$$\omega_m(f, t)_p \leqslant t\omega_{m-1}(f', t)_p \leqslant \cdots \leqslant \begin{cases} t^k \omega_{m-k}(f^{(k)}, t)_p, & m > k; \\ t^m \|f^{(m)}\|_p = t^m \omega_0(f^{(m)}, t)_p, & m \leqslant k. \end{cases} \tag{1.1}$$

The inverse of (1.1) with any constant independent of f is not true in general (see [8, Theorem 4] for a counterexample).

Yu and Zhou [17] proved part of the inverse in a special case, namely,

$$h\omega_{r-1}(S', h)_\infty \leqslant C_r \omega_r(S, h)_\infty, \tag{1.2}$$

where S is any spline of order r with equally spaced knots, and h is the mesh size. Hu and Yu [9] proved that for such splines S the whole inverse of (1.1) holds true, thus $\omega_m(S, t)_p \sim t\omega_{m-1}(S', t)_p \sim t^2\omega_{m-2}(S'', t)_p \cdots$. In this paper we generalize it to splines with arbitrary (fixed) knots, and further to shift-invariant spaces and wavelets under certain conditions. We discuss the case of splines in Section 2, and extend the results further to shift-invariant spaces and wavelets in Section 3. Applications will be given in the last section.

2. SPLINES

Let $r > 0$ be an integer and let $\mathbf{T} := \{x_i\} \subset \mathrm{Int}(A)$ with $x_i < x_{i+r}$ be a given non-decreasing knot sequence. $\mathbf{T}$ is not allowed to have finite cluster points. If either endpoint of A is finite, we need to add r auxiliary knots outside A to support a B-spline basis of order r. If A has a finite left endpoint a and $x_1 > a$ is the first knots in $\mathbf{T}$, we choose, without loss of generality, $x_j := a + j(x_1 - a)$, $j = 0, -1, ..., -r+1$, as auxiliary knots to the left. If A has a finite right endpoint b and $x_n < b$ is the last knot in $\mathbf{T}$, we similarly choose $x_j := b + (j - n - 1)(b - x_n)$, $j = n+1, ..., n+r$. If the right (left) endpoint of A is $\infty(-\infty)$, we require $\lim_{i\to\infty} x_i = \infty$ ($\lim_{i\to-\infty} x_i = -\infty$), respectively. Therefore we can write $\mathbf{T} = \{x_i\}_{i\in\Lambda}$, where $\Lambda = [-r+1, n+r] \cap \mathbf{Z}$ ($\mathbf{Z}$ is the set of all integers) if $A = [a, b]$; $\Lambda = \mathbf{Z}$ if $A = (-\infty, \infty)$; $\Lambda = (-\infty, n+r] \cap \mathbf{Z}$ if $A = (-\infty, b]$; and $\Lambda = [-r+1, \infty) \cap \mathbf{Z}$ if $A = [a, \infty)$.

We shall use the notation $I_i := [x_i, x_{i+1}]$, and $\tilde{I}_i := [x_{i-r+1}, x_{i+r}] \cap A$. The mesh size of $\mathbf{T}$ is denoted by $\bar{\delta} := \max\{|I_i|\}$, where $|I_i| := x_{i+1} - x_i$ is the measure of I_i, and the length of the shortest subinterval by $\underline{\delta} := \min\{|I_i| : x_{i+1} > x_i\}$. The space of all splines of order r on $\mathbf{T}$ is denoted by $\mathcal{S}_r = \mathcal{S}_r(\mathbf{T}) = \mathcal{S}_r(\mathbf{T}, A)$, and the B-splines of order r on knot sequence $\mathbf{T}$ by

$$N_{ir}(x) := N(x; x_i, ..., x_{i+r}) := (x_{i+r} - x_i)[x_i, ..., x_{i+r}](\cdot - x)_+^{r-1}.$$

It is well known that these B-splines form a basis $\mathcal{S}_r$, and every spline $S \in \mathcal{S}_r$ can be written as

$$S = \sum_i c_i N_{ir}, \tag{2.1}$$

where i runs from $-r+1$ to n if $A = [a, b]$, from $-\infty$ to n if $A = (-\infty, b]$, and so on. The difference operators on finite or infinite sequences of real numbers $\mathbf{c} := \{c_i\}$ are defined by

$$\Delta^k c_i := \Delta(\Delta^{k-1} c_i) = \sum_{j=0}^{k} (-1)^{k+j} \binom{k}{j} c_{i+j}, \qquad k \geqslant 1,$$

and $\Delta^k \mathbf{c} := \{\Delta^k c_i\}$. When these operators are applied to the coefficient sequence of (2.1), the largest admissible i is $n-k$ if A has a finite right endpoint b. We shall use a discrete norm $|||\mathbf{c}|||_p := \|\mathbf{c}'\|_{l_p}$, where $\mathbf{c}' := \{c_i'\}$, $c_i' := d_i^{1/p} c_i$, and $d_i := (x_{i+r} - x_i)/r$. With the notation above, the main theorem in Hu and Yu [9] can be stated as

THEOREM A. *Let* $\mathbf{T}$ *be an equally spaced knot sequence with* $h := |I_i| = \bar{\delta}$, S *be such as in* (2.1), *and let* $0 \leqslant t \leqslant h$. *If* $m < r$, *then*

$$t^j \omega_{m-j}(S^{(j)}, t)_p \sim \left(\frac{t}{h}\right)^m |||\Delta^m \mathbf{c}|||_p, \qquad 0 \leqslant j \leqslant m, \tag{2.2}$$

therefore they are all equivalent. For $m \geqslant r$ *we have*

$$t^j \omega_{m-j}(S^{(j)}, t)_p \sim \left(\frac{t}{h}\right)^{r-1+1/p} |||\Delta^r \mathbf{c}|||_p, \qquad 0 \leqslant j < r, \tag{2.3}$$

and again they are all equivalent. Moreover, the equivalence constants depend only on $\max(r, m)$ *in either case.*

This theorem includes (1.2) as a special case, and also generalizes in another direction the following theorem for splines with arbitrary knot sequence by de Boor (see [1] and [14] for $p = \infty$, and [6] for $1 \leqslant p < \infty$).

THEOREM B. *Let S be as in* (2.1). *Then $\|S\|_p$ and $|||\mathbf{c}|||_p$ are equivalent,*

$$D_r\,|||\mathbf{c}|||_p \leqslant \|S\|_p \leqslant |||\mathbf{c}|||_p,$$

where $D_r > 0$ is a constant depending only on r.

We now remove the requirement of equal spacing in Theorem A by proving the following two theorems.

THEOREM 1. *Suppose $0 \leqslant t \leqslant \bar{\delta}$. If the multiplicity of every knot in $\mathbf{T}$ is no greater than $r-m$ for some $0<m<r$, that is, if $\mathcal{S}_r \subset \mathbf{W}_p^m$, then for any $S \in \mathcal{S}_r$*

$$t^j \omega_{m-j}(S^{(j)}, t)_p \sim (t/\bar{\delta})^m\, \omega_m(S, \bar{\delta})_p, \qquad 0 \leqslant j \leqslant m, \tag{2.4}$$

where the equivalence constants depend on r and the ratio $\bar{\delta}/\underline{\delta}$.

THEOREM 2. *Let $m \geqslant r$, $\lambda := r-1+1/p$, and $0 \leqslant t \leqslant \bar{\delta}$. If all interior knots in $\mathbf{T}$ are single, then*

$$t^j \omega_{m-j}(S^{(j)}, t)_p \sim t^\lambda \left(\sum_i |\mathcal{J}_i|^p\right)^{1/p}, \qquad 0 \leqslant j < r, \tag{2.5}$$

where $\mathcal{J}_i$ is the jump of $S^{(r-1)}$ at x_i, and the equivalence constants depend on m and the ratio $\bar{\delta}/\underline{\delta}$.

Remark. The equivalence in (2.4) and (2.5) also holds for all t greater than and comparable with $\bar{\delta}$. When t is large, however, there is no equivalence among $t^j \omega_{m-j}(S^{(j)}, t)_p$ unless we allow C to depend on the ratio $t/\bar{\delta}$. See [9] for a counterexample in $\mathbf{C}$. For general case, consider $t=1$ and $S = \sum_i (-1)^i N_{ir}$ on $[0, 1]$ with n equally spaced interior knots. One can see that $\omega_{m-1}(S', 1)_p$ increases with n but $\omega_m(S, 1)_p$ does not.

The proofs turned out to be easy, to our great surprise. But one can no longer measure the moduli by discrete norms $|||\cdot|||_p$ in terms on B-spline coefficients, at least not with our proofs. We need some lemmas for the proofs; the first of them can be found in DeVore and Lorentz's book [6, Proposition 5.4.6].

LEMMA C. *If the spline $S := Q_{\mathbf{T}}(f)$, where $Q_{\mathbf{T}}$ is the quasi-interpolant defined in* [6, *Section* 5.4], *then for $1 \leqslant m < r$*

$$|S^{(m)}(x)| \leqslant C_r |I_i|^{-m-1/p}\, E_m(f, \tilde{I}_i)_p, \qquad x \in (x_i, x_{i+1}), \tag{2.6}$$

where $E_m(f, I)_p$ is the degree of approximation of f on I by algebraic polynomials of degree $<m$.

LEMMA 2.1. *Let $0 \leqslant j \leqslant m < r$, and $S \in \mathcal{S}_r \cap \mathbf{W}_p^m$. Then*

$$\left\{\sum_i (|I_i|^{m-j} \|S^{(m)}\|_{\mathbf{L}_p(L_i)})^p\right\}^{1/p} \leqslant C_r \omega_{m-j}(S^{(j)}, A, \bar{\delta})_p. \tag{2.7}$$

Proof. If $x_{i+1} > x_i$, by Lemma C and the fact that the quasi-interpolant $Q_{\mathbf{T}}$ is a projection from $\mathbf{L}_p$ to $\mathcal{S}_r$, that is, $S = Q_{\mathbf{T}}(S)$, we have

$$|S^{(m)}(x)| \leqslant C\, |I_i|^{-m-1/p}\, E_m(S, \tilde{I}_i)_p, \qquad x \in (x_i, x_{i+1}). \tag{2.8}$$

If we view $S^{(m)}$ as $(S^{(j)})^{(m-j)}$, and note that $S^{(j)} \in \mathbf{W}_p^{m-j}$, this becomes

$$|S^{(m)}(x)| \leqslant C|I_i|^{-m+j-1/p}\, E_{m-j}(S^{(j)}, \tilde{I}_i)_p, \qquad x \in (x_i, x_{i+1}).$$

Integrate this over I_i we derive

$$|I_i|^{m-j}\, \|S^{(m)}\|_{\mathbf{L}_p(I_i)} \leqslant CE_{m-j}(S^{(j)}, \tilde{I}_i)_p \leqslant C\omega_{m-j}(S^{(j)}, \tilde{I}_i, |\tilde{I}_i|)_p. \tag{2.9}$$

And this is also (trivially) true for the case of $x_i = x_{i+1}$. Raising both sides of (2.9) to the pth power and adding over i give

$$\sum_i (|I_i|^{m-j}\, \|S^{(m)}\|_{\mathbf{L}_p(I_i)})^p \leqslant C^p \sum_i \omega_{m-j}(S^{(j)}, \tilde{I}_i, |\tilde{I}_i|)_p^p \leqslant C^p \omega_{m-j}(S^{(j)}, A, \bar{\delta})_p^p.$$

For the second inequality above, see [12] and [7] for the case of a finite interval $[a, b]$. The general case can be easily proved through use of an average modulus of smoothness, cf. [6, Section 6.5]. ∎

LEMMA 2.2. *Let S be any spline in $\mathcal{S}_r \cap \mathbf{W}_p^m$ for some $m < r$. Then*

$$\bar{\delta}^j\, \omega_{m-j}(S^{(j)}, \bar{\delta})_p \sim \omega_m(S, \bar{\delta})_p \qquad 0 \leqslant j \leqslant m, \tag{2.10}$$

where the equivalence constants depend on r and the ratio $\bar{\delta}/\underline{\delta}$.

Proof. Lemma 2.1 with $j = 0$ gives

$$\underline{\delta}\, \|S^{(m)}\|_{\mathbf{L}_p(A)} \leqslant C\left\{\sum_i (|I_i|^m\, \|S^{(m)}\|_{\mathbf{L}_p(I_i)})^p\right\}^{1/p} \leqslant C\omega_m(S, A, \bar{\delta})_p,$$

and (2.10) easily follows from this and (1.1). ∎

Proof of Theorem 1. For any $0 < t \leqslant \bar{\delta}$, we have

$$\begin{aligned}\omega_m(S, \bar{\delta})_p = \omega_m(S, (\bar{\delta}/t)\, t)_p &\leqslant C(\bar{\delta}/t)^m\, \omega_m(S, t)_p \leqslant C(\bar{\delta}/t)^m\, t\omega_{m-1}(S', t)_p \\ &\leqslant C(\bar{\delta}/t)^m\, t^2\omega_{m-2}(S'', t)_p \leqslant \cdots \leqslant C(\bar{\delta}/t)^m\, t^m\, \|S^{(m)}\|_p \\ &= C\bar{\delta}^m\, \|S^{(m)}\|_p \leqslant C\omega_m(S, \bar{\delta})_p,\end{aligned}$$

which gives (2.4). ∎

Proof of Theorem 2. The proof is almost identical to that of (1.9) in Hu and Yu [9, Theorem 2]. The difference is, of course, that the spline s in [9] has equal spacing. Note that, however, their proof does not use equal spacing of s, but that of the B-splines $\tilde{N}_i$ defined in (2.10–11) of [9], which is introduced by the difference operator Δ_t^m. As long as $mt \leqslant \underline{\delta}$, their arguments are valid here, and can be copied almost line by line. ∎

3. SHIFT-INVARIANT SPACES AND WAVELETS

We now extend our results to *shift-invariant* spaces (SI) and wavelets. We shall use some notation and properties very common in the literature without explicitly mentioning any references, most of them can be found in, for example, [2, 5, 11, 3, 4, 13]. Although many of the known results we mention here are true for $0 < p \leqslant \infty$, we shall concentrate on the case $1 \leqslant p \leqslant \infty$, since our results are only true for $p \geqslant 1$. In this section, we prove our main result Theorem 3 only on $A = \mathbf{R}$, but it should be pointed out that it holds true if we restrict every function involved to any interval. A space $\mathscr{S}$ of functions defined on $\mathbf{R}$ is said to be shift-invariant if

$$f \in \mathscr{S} \Leftrightarrow f(\cdot + i) \in \mathscr{S} \qquad \text{for all} \quad i \in \mathbf{Z}.$$

In other words, $\mathscr{S}$ contains all integer translates of f if it contains f. One of the simplest SI spaces, called *principal shift-invariant* space (PSI), is generated by a single function φ:

$$\mathscr{S} = \mathscr{S}(\varphi) := \Big\{S = \sum_{i \in \mathbf{Z}} c_i \varphi(\cdot - i) \colon \{c_i\} \in l_p(\mathbf{Z})\Big\}. \tag{3.1}$$

We make the following assumptions about φ.

1. φ is supported on an interval $[-\rho, \rho]$, with ρ a positive integer;
2. $\varphi \in \mathbf{W}_p^k(\mathbf{R})$ for some $0 < k \leqslant r$;

3. φ satisfies Strang–Fix conditions for some positive integer r:
   (i) $\hat{\varphi}(0) = 1$;
   (ii) $D^{\nu}\hat{\varphi}(2\pi i = 0,\ i \in \mathbf{Z}\backslash\{0\},\ 0 \leqslant \nu < r$;
4. the functions $\varphi_i := \varphi(\cdot - i)$, $i \in \mathbf{Z}$, are globally linearly independent.

For any $h > 0$, the scaled spaces $\mathcal{S}_h$ of $\mathcal{S}$ is defined by

$$\mathcal{S}_h := \mathcal{S}_h(\varphi) := \{S(\cdot/h) \colon S \in \mathcal{S}\}.$$

In applications, one is interested in how well a general function f is approximated by elements of $\mathcal{S}_h$. From Assumption 4 we know that any $S_h \in \mathcal{S}_h(\varphi)$ can be uniquely written as

$$S_h = \sum_{i \in \mathbf{Z}} c_i \varphi_i(\cdot/h) = \sum_{i \in \mathbf{Z}} c_i \varphi(\cdot/h - i), \qquad \{c_i\} \in l_p(\mathbf{Z}). \tag{3.2}$$

The series converges uniformly on any compact set since for any $x \in \mathbf{R}$, there are at most 2ρ nonzero terms in $S_h(x)$, and it has been shown in the literature that it converges in the $\mathbf{L}_p$ topology, too. It is well-known that Strang–Fix conditions imply that any polynomial in $\mathbf{P}_{r-1}$, the space of all polynomials of degree $< r$, is contained in $\mathcal{S}(\varphi)$ locally.

Closely related to PSI are wavelets. Together with a wavelet φ we shall use its dyadic dilates $\varphi(\cdot 2^n)$, $n \in \mathbf{Z}$, and their translates $\varphi_{ni} := \varphi(\cdot 2^n - i)$, $n, i \in \mathbf{Z}$. If $n = 0$, we omit n in the notation: $\varphi_i := \varphi_{0i} = \varphi(\cdot - i)$. In addition to the four assumptions above, we further assume that φ satisfies the refinement equation:

5. $\varphi = \varphi_0 = \sum_i a_i \varphi_{1i}$, where the sum is taken over a finite number of $i \in \mathbf{Z}$.

It turns out that the summation only contains those i for which supp $\varphi_{1i} \subset$ supp φ. It has been shown that any $f \in \mathbf{L}_p$ has wavelet decomposition

$$f = \sum_{l \in \mathbf{Z}} \sum_{i \in \mathbf{Z}} a_{li} \varphi_{li} \tag{3.3}$$

with convergence in $\mathbf{L}_p$. In applications, one often compresses (say, choosing a finite number of terms) or truncates the decomposition. Suppose that the highest resolution after this action is n, one can use the refinement equation to rewrite the result as

$$S_n = \sum_{i \in \mathbf{Z}} c_i \varphi_{ni} = \sum_{i \in \mathbf{Z}} c_i \varphi(\cdot 2^n - i). \tag{3.4}$$

Since (3.4) is a special case of (3.2) with $h=2^{-n}$, we shall concentrate on (3.2) in the rest of the section, and show the inverse of (1.1) holds for functions in that form. Namely we prove

THEOREM 3. *Let S_h be as in (3.2). Then for any $0 \leqslant t < h$ and $m \leqslant k$, all quantities*

$$t^j \omega_{m-j}(S_h^{(j)}, t)_p, \qquad 0 \leqslant j \leqslant m, \tag{3.5}$$

are equivalent, with equivalence constants depending only on φ and m.

Proof. If we let $y := h^{-1}x, u := h^{-1}t$, and $F(y) := f(hy) = f(x)$, then $t^{j-1/p}\omega_{m-j}(f^{(j)}, t)_p = u^{j-1/p}\omega_{m-j}(F^{(j)}, u)_p$. Therefore we only need to prove (3.5) for $h=1$.

We first prove an analogue of (2.6) (and (2.8)) by mimicking its proof in DeVore and Lorentz's book [6]. Let c_v be the dual functionals to the basis $\varphi(\cdot - i)$ of $\mathscr{S} := \mathscr{S}_1$, that is, $c_v(\varphi_i) = \delta_{iv}$, then each $S \in \mathscr{S}$ can be written as

$$S = \sum_i c_i(S)\, \varphi_i.$$

Note that $c_i(S) = c_0(S(\cdot + i))$ and the norm of c_0 as an operator in $\mathbf{L}_p$ depends only on φ:

$$|c_0(S)| \leqslant C\, \|S\|_{\mathbf{L}_p(I_\mu)}, \qquad S \in \mathscr{S},$$

for any integer μ such that φ does not vanish identically on $I_\mu = [\mu, \mu+1]$. Since φ is compactly supported, there are only 2ρ such intervals. We now fix an interval $I_v = [v, v+1]$. Since $\mathbf{P}_{r-1} \subseteq \mathscr{S}$ locally, for any polynomial $P \in \mathbf{P}_{m-1} \subseteq \mathbf{P}_{r-1}$ and $\forall x \in I_v$, we have

$$S(x) - P(x) = \sum_{i=v+1-\rho}^{v+\rho} c_i(S-P)\, \varphi_i(x)$$

and

$$\begin{aligned} |S^{(m)}(x)| = |(S-P)^{(m)}(x)| &\leqslant \sum_{i=v+1-\rho}^{v+\rho} |c_i(S-P)\, \varphi_i^{(m)}(x)| \\ &\leqslant C\, \|S-P\|_{\mathbf{L}_p(I_v)} \sum_{i=v+1-\rho}^{v+\rho} |\varphi_i^{(m)}(x)|. \end{aligned}$$

Raising both side of the inequality to the pth power and integrating over I_v, we obtain

$$\|S^{(m)}\|_{\mathbf{L}_p(I_v)} \leqslant C\,\|\varphi^{(m)}\|_{\mathbf{L}_p(\mathbf{R})}\,\|S-P\|_{\mathbf{L}_p(I_v)} \leqslant \|S-P\|_{\mathbf{L}_p(I_v)},$$

here in the last step we are able to drop $\|\varphi^{(m)}\|_{\mathbf{L}_p(\mathbf{R})}$ because we allow C to depend on φ and m. Take minimum of this over all polynomials $P \in \mathbf{P}_{m-1}$, we obtain the desired analogue

$$\|S^{(m)}\|_{\mathbf{L}_p(I_v)} \leqslant CE_m(S, I_v)_p \leqslant C\omega_m(S, I_v, 1)_p.$$

If we view $S^{(m)}$ as $(S^{(j)})^{(m-j)}$, this becomes

$$\|S^{(m)}\|_{\mathbf{L}_p(I_v)} \leqslant CE_{m-j}(S^{(j)}, I_v)_p \leqslant C\omega_{m-j}(S^{(j)}, I_v, 1)_p,$$

which corresponds to (2.9), (we remind the reader that $|I_v| = 1$). The rest of the proof is similar to that of Lemma 2.1. ∎

4. APPLICATIONS

In this section, we give some possible applications of our results. We first estimate derivatives of spline and PSI approximants. We point out that although estimates for wavelet compressions on $\mathbf{R}$ similar to the following examples may not be very useful due to very small values of $h = 2^{-n}$, nothing can prevent one from using the results in an area where the function f is relatively flat hence n relatively small.

Example 1. Let $r > 3$ and $2 \leqslant k < r$. Let $\mathbf{T}$ be a knot sequence with $\bar{\delta}$ comparable with n^{-1}, and S a spline in $\mathscr{S}_r(\mathbf{T}) \cap \mathbf{W}_p^3$. If S approximates a function $f \in \mathbf{W}_p^3$ with an error

$$\|f - S\|_p \leqslant C_0 n^{-1} \omega_k(f', n^{-1})_p, \tag{4.1}$$

where $C_0 \geqslant 1$, and we want to estimate $\|S^{(3)}\|_p$, then

$$\begin{aligned}\omega_3(S, n^{-1})_p &\leqslant \omega_3(f - S, n^{-1})_p + \omega_3(f, n^{-1})_p \leqslant C\,\|f - S\|_p + n^{-1}\omega_2(f', n^{-1})_p \\ &\leqslant CC_0 n^{-1}(\omega_k(f', n^{-1})_p + \omega_2(f', n^{-1})_p) \leqslant CC_0 n^{-1}\omega_2(f', n^{-1})_p.\end{aligned}$$

From $n^{-3}\|S^{(3)}\|_p \sim \omega_3(S, n^{-1})_p$, we obtain

$$\|S^{(3)}\|_p \leqslant CC_0 n^2 \omega_2(f', n^{-1})_p \leqslant CC_0 n\omega(f'', n^{-1})_p \leqslant CC_0\,\|f^{(3)}\|_p,$$

with C depending on r, $n\bar{\delta}$, and $\bar{\delta}/\underline{\delta}$.

The assumption $k \geqslant 2$ in Example 1 is not essential for estimating $\|S^{(3)}\|_p$. Even if (4.1) is replaced by a lower order Jackson inequality, one can still estimate the size of $S^{(3)}$, or any higher order derivative of S, in a different format, of course. We illustrate this in a second example.

EXAMPLE 2. Let $k \geqslant 0$, and let $f \in \mathbf{L}_p$ be approximated by an element S_h of a PSI space $\mathscr{S}_h(\varphi) \subset \mathbf{W}_p^3$ with

$$\|f - S_h\|_p \leqslant C\omega_k(f, h)_p. \tag{4.2}$$

By the subadditivity of ω_m and the fact $\omega_m(g, t)_p \leqslant C_m \omega_{m-1}(g, t)_p$ for any $g \in \mathbf{L}_p$ we derive

$$\omega_3(S_h, h)_p \leqslant \omega_3(f - S_h, h)_p + \omega_3(f, h)_p \leqslant C\omega_{\tilde{k}}(f, h)_p.$$

where $\tilde{k} := \min(k, 3)$. From $h^3 \|S_h^{(3)}\|_p \sim \omega_3(S_h, h)_p$ we conclude

$$\|S_h^{(3)}\|_p \leqslant Ch^{-3}\omega_{\tilde{k}}(f, h)_p, \tag{4.3}$$

with the constant C depending only on φ.

There are examples in [8, Theorem A] and [9] on how (special cases of) our theorems can be use to "transplant" results on degree of approximation by one kind of approximants to that by another kind. With the generalized results in this paper, some restrictions (such as equal spacing for splines) can be removed, of course. Here we give one more example, in which Shvedov's counterexample on convex polynomial approximation in [16] is transplanted to convex spline approximation. Suppose for any $n \geqslant r+3$ there is a knot sequence containing n interior knots: $\mathbf{T}_n = \{x_{ni}\}_{i=-r+1}^{n+r}$ (with auxiliary knots, see the beginning of Section 2), on $[0, 1]$. We define its mesh size by $\bar{\delta}_n := \max_i\{|I_{ni}|\}$, where $I_{ni} := [x_{ni}, x_{n,i+1}]$, and its length of the shortest subinterval by $\underline{\delta}_n := \min_i\{|I_{ni}|: x_{n,i+1} > x_{ni}\}$.

THEOREM 4. *Let $r \geqslant 3$ be an integer, and let $\mathbf{T}_n$, $n = r+3, r+4, ...,$ be any knot sequences such that both $(n\bar{\delta}_n)^{-1}$ and $\bar{\delta}_n/\underline{\delta}_n$ are bounded by an absolute constant $M > 0$, and that $\mathscr{S}_r^n := \mathscr{S}_r(\mathbf{T}_n, [0, 1]) \subset \mathbf{W}_\infty^2[0, 1]$. Then for any $K > 0$ and $n \geqslant r+3$ there exists a convex function $f \in \mathbf{C}[0, 1]$ such that*

$$E^{(2)}(f, \mathscr{S}_r^n)_\infty \geqslant K\omega_4(f, n^{-1})_\infty, \tag{4.4}$$

where $E^{(2)}(f, \mathscr{S}_r^n)_\infty$ is the degree of approximation of f by convex splines from $\mathscr{S}_r^n$.

Proof. Suppose (4.4) is not true, that is, if for every convex function $f \in \mathbf{C}[0, 1]$, there is a convex spline $S \in \mathcal{S}_r^n$ such that

$$\|f - S\|_\infty \leqslant K\omega_4(f, n^{-1})_\infty,$$

then we can apply a result by Manya and Shevchuk (see [10] and [15]) to this S and obtain a convex polynomial P_n such that

$$\|S - P_n\|_\infty \leqslant C_1 n^{-2} \omega_2(S'', n^{-1})_\infty < C_2 \omega_4(S, n^{-1})_\infty \leqslant C_3 \omega_4(f, n^{-1})_\infty, \tag{4.5}$$

with C_3 depending only on M and K. Thus

$$\|f - P_n\|_\infty \leqslant \|f - S\|_\infty + \|S - P_n\|_\infty \leqslant (C_3 + K)\, \omega_4(f, n^{-1})_\infty,$$

which contradicts Shvedov's counterexample. ∎

In fact, as pointed out by a referee of this paper, one can show (4.4) with n^{-1} replaced by any $0 < \eta \leqslant 1$ for any fixed knot sequence. We state this in the last theorem of the paper.

THEOREM 5. *Let $r \geqslant 3$ be an integer, and let $\mathbf{T} = \{x_i\}_{i=-r+1}^{N+r}$ be any knot sequence on $[0, 1]$ as described in Section 2 such that $\mathcal{S}_r(\mathbf{T}, [0, 1]) \subset \mathbf{W}^2_\infty[0, 1]$. Then*

$$\sup_{f \in \mathbf{U}} \frac{E^{(2)}(f, \mathcal{S}_r)_\infty}{\omega_4(f, 1)_\infty} = \infty, \tag{4.6}$$

where $\mathbf{U}$ denotes the set of all convex functions in $\mathbf{C}[0, 1] \backslash \mathbf{P}_3$.

Proof. The statement (4.6) is equivalent to that for any $K > 0$, $\exists f \in \mathbf{U}$ such that

$$E^{(2)}(f, \mathcal{S}_r)_\infty \geqslant K\omega_4(f, 1)_\infty, \tag{4.7}$$

which can be shown just like (4.4) except that we have to choose the degree of the polynomial P_n in (4.5) so that $n \geqslant \max(\bar{\delta}^{-1}, r+3)$ in order to apply Theorem 1 and Manya and Shevchuk's result. ∎

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REFERENCES

1. C. de Boor, "Practical Guide to Splines," Springer-Verlag, New York, 1987.
2. C. de Boor, R. A. DeVore, and A. Ron, Approximation from shift-invariant subspaces of $\mathbf{L}_2(\mathbf{R}^d)$, *Trans. Amer. Math. Soc.* **341** (1994), 787–806.
3. C. K. Chui, "An Introduction to Wavelets," Academic Press, San Diego, 1992.
4. I. Daubechies, "Ten Lectures on Wavelets," SIAM, Philadelphia, 1992.
5. R. A. DeVore, B. Jawerth, and V. Popov, Compression of wavelet decompositions, *Amer. J. Math.* **114** (1992), 737–785.
6. R. A. DeVore and G. G. Lorentz, "Constructive Approximation," Springer-Verlag, Berlin, 1993.
7. Y. K. Hu, Positive and copositive spline approximation in $\mathbf{L}_p[0, 1]$, *Comput. Math. Appl.* **30** (1995), 137–146.
8. Y. K. Hu, D. Leviatan, and X. M. Yu, Positive polynomial and spline approximation, *J. Approx. Theory* **80** (1995), 204–218.
9. Y. K. Hu and X. M. Yu, Discrete modulus of smoothness of splines with equally spaced knots, *SIAM Numer. Anal.* **32** (1995), 1428–1435.
10. K. A. Kopotun, Pointwise and uniform estimates for convex approximation of functions by algebraic polynomials, *Constr. Approx.* **10**, No. 2 (1994), 153–178.
11. G. Kyriazis, Approximation from shift-invariant spaces, *Constr. Approx.* **11** (1995), 141–164.
12. D. Leviatan and H. N. Mhaskar, The rate of monotone spline approximation in the $\mathbf{L}_p$ norm, *SIAM J. Math. Anal.* **13** (1982), 866–874.
13. Y. Meyer, "Wavelets; Algorithms & Applications," SIAM, Philadelphia, 1993. [transl. and rev. by R. D. Ryan]
14. L. L. Schumaker, "Spline Functions: Basic Theory," Wiley, New York, 1981.
15. I. A. Shevchuk, "Approximation by Polynomials and Traces of the Functions Continuous on an Interval," Naukova Dumka, Kiev, 1992. [In Russian]
16. A. S. Shvedov, Orders of coapproximation of functions by algebraic polynomials, *Mat. Zametki* **29**, No. 1 (1981), 117–130 [in Russian]; *Math. Notes* **29**, No. 1 (1981), 63–70. [Engl. transl.]
17. X. M. Yu and S. P. Zhou, On monotone spline approximation, *SIAM J. Math. Anal.* **25** (1994), 1227–1239.